Hidden Inscriptions in the Laurentian Library

Paul L. Rosin and Ralph R. Martin
Department of Computer Science
Cardiff University
UK
{Paul.Rosin,Ralph.Martin}@cs.cf.ac.uk

Abstract

Following the discovery of a series of geometric designs on the floor of the Biblioteca Laurenziana in Florence we analyse the "Medici panel", which consists of a circular rosette containing a set of inscribed elliptical disks, with a view to regenerating the pattern. The geometry is shown to be surprisingly complex. We were able to determine a numerical solution for synthesising the pattern. Alternatively we found analytic solutions for simpler patterns such as inscribed circles or polygonal rosettes.

1 Introduction

In 1774 a dramatic event occurred in the reading room (see Figure 1) of the Biblioteca Laurenziana in Florence, when a desk overloaded with books collapsed. During the repairs it was discovered that underneath the wooden floorboards supporting the reading desks lay a pavement containing a series of terracotta panels, each with a different geometric design (see Figure 3).

To explain the mystery of why these carefully designed and executed works had been hidden we need to look into the history of the building's construction. The library was designed by Michelangelo for the Medici Pope Clemente VII to store the Medici's collection of books and manuscripts. Although work was started in 1524 there were many delays and alterations, and when it finally opened (37 years after Michelangelo leaving Florence and 7 years after his death) in 1571, it was still unfinished! It appears that Michelangelo's original plan for a central aisle of desks was vetoed by the Pope since more seating and storage were required. This necessitated the modified version finally executed which has two side aisles of desks, unfortunately covering the geometric panels. The inlaid pattern in the central aisle was then designed and constructed shortly afterwards (1549-1554) by Santi Buglioli and Tribolo, following the carved ceiling by Battista del Tasso and Antonio Carota (1534) which was based in turn on designs by Michelangelo. Unfortunately, while some features of the library such as these are well documented, little is known about the hidden floor panels.[1] Moreover, due to their remaining covered up to this day, their mere existence is still not well known. Even standard works give no information [1, 12] or just include a fleeting mention [5]. However, since the 1980s Ben Nicholson has been investigating many aspects (including philosophical, psychological, and aesthetical themes as well as the geometric) of the panels [8, 9].

This paper will look more closely at the geometry behind the "Medici panel" which is illustrated in Figure 2. This shows a circular rosette, similar in form to the type much favoured in architecture since Roman times [11]. Many detailed and subtle aspects of its construction have been carefully analysed by Nicholson [8, 9] (see also Kappraff [7]). For instance, it is not perfectly square, but actually has an aspect ratio of 12:13. Nicholson shows that this is an important aspect of the shape as the lengths of the diagonals of both the inscribed square and rectangle are used to lay out geometric figures as part of the construction process. In addition, the difference between the diagonals provides a means for determining the spacing between the curvilinear rhombuses. We will call these rhombuses in the patterns "slots". The particular feature we are interested in is the unusual occurrence of the inset ovals within the slots. While Nicholson described many aspects of the rosette's construction, he stops short at these ovals. In this paper we will attempt to see why.

[1] It should be mentioned that, perhaps unsurprisingly in light of this sparsity of information, there is not total agreement that Michelangelo was involved in the design of the panels (J. Ackermann – personal communication).

Figure 1: The reading room of the Biblioteca Laurenziana

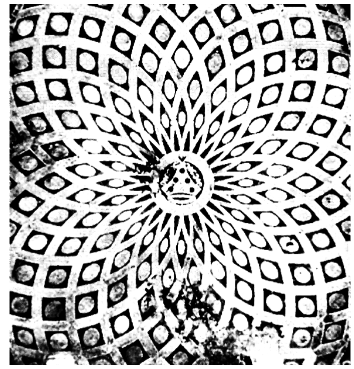


Figure 2: The “Medici panel” – a circular rosette.

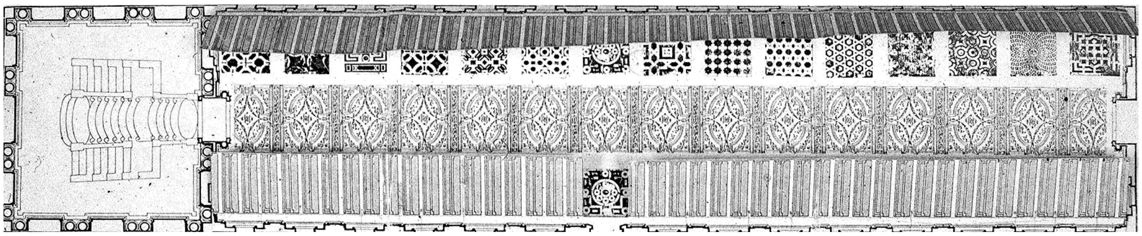


Figure 3: This montage (from Nicholson [8]) shows the library with the geometric panels revealed.

2 Finding the ellipse

The Biblioteca Laurenziana uses a pair of rosettes slightly rotated against each other to produce the bands which therefore expand as they radiate out from the centre. To simplify the problem we shall only treat the rosette’s bands as infinitely thin circles. We will also assume that the ellipses are tangent to these circles rather than yet another set of implicit circles offset slightly from the bands (see Figure 4).

2.1 Problem statement

The circular rosette we are considering is made up of a set of n circles of equal radius, which are distributed about the centre of the rosette at even angular increments of $\frac{2\pi}{n}$ radians, and all pass though the rosette’s centre.[2] Here we outline our approach to finding the parameters of the family of ellipses (see Figure 5) which form a tile,

[2] In practice, artists often make use of additional reference circles: the outer envelope and an inner “centrum” ring; see for instance Dürer’s description [4].

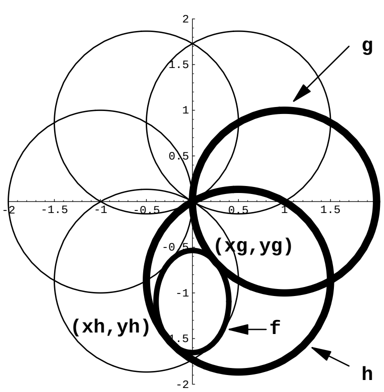


Figure 4: A circular rosette containing $n = 6$ circles. An ellipse (f) is shown tangent to two adjacent circles (g and h). Also indicated are the two contact points (x_g, y_g) and (x_h, y_h).

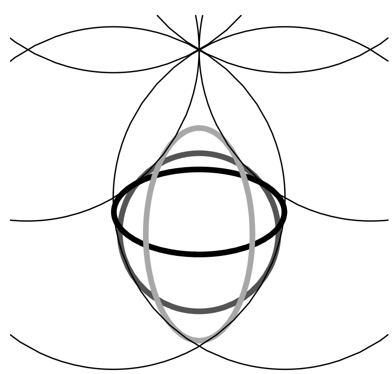


Figure 5: Three possible ellipses from the family that can be inscribed in the same slot.

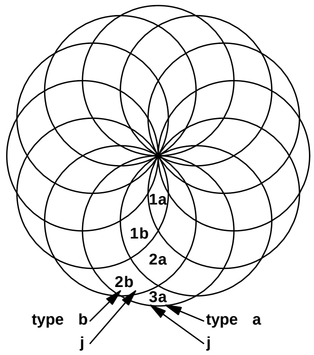


Figure 6: Notation for slots in the rosette

n	type	θ_g	θ_h
even	a	$(4j-4)\pi/(2n)$	$4j\pi/(2n)$
even	b	$(4j-2)\pi/(2n)$	$(4j+2)\pi/(2n)$
odd	a	$(4j-5)\pi/(2n)$	$(4j-1)\pi/(2n)$
odd	b	$(4j-3)\pi/(2n)$	$(4j+1)\pi/(2n)$

Table 1: Positions of defining circles

fitting in between the main circles generating the pattern.

Let us consider a single ellipse from the family of ellipses which it is possible to put into a single slot in the pattern. To keep the mathematics simple, but without loss of generality, we assume the defining circles have a radius of 1 unit, and that the centre of the ellipse to be found lies on the y axis. The ellipse can thus be written as

$$f(x, y; a, b, y_0) = \frac{x^2}{a^2} + \frac{(y - y_0)^2}{b^2} - 1 = 0.$$

The centre of the ellipse is at a height y_0 and its horizontal and vertical axis lengths are $2a$ and $2b$. Because a single parameter family of ellipses fits in the pattern in a given slot, if we choose say y_0, this determines a and b. For simplicity in the derivation, we replace a^2 by A, b^2 by B, and then replace B by SA where S measures the vertical elongation of the ellipse is. The defining circles (see Figure 4) are at

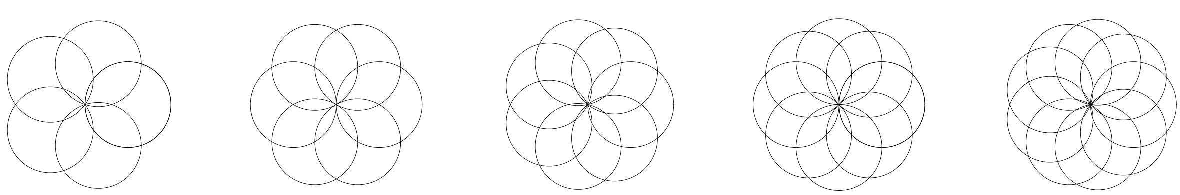
$$\begin{aligned} g(x, y) &= (x - x_g)^2 + (y - y_g)^2 - 1 = 0 \\ h(x, y) &= (x - x_h)^2 + (y - y_h)^2 - 1 = 0, \end{aligned}$$

where

$$\begin{aligned} x_g &= \cos(\theta_g), \qquad & y_g &= \sin(\theta_g) \\ x_h &= \cos(\theta_h), \qquad & y_h &= \sin(\theta_h). \end{aligned}$$

We let n be the number of defining circles, while j denotes the ellipse's slot in the pattern, as shown by the numbering in Figure 6. The values for θ_g and θ_h depend on whether n is even or odd, and also on whether we want "type (a)" or "type (b)" ellipses—see Figure 6. Thus, we must consider 4 cases as shown in Table 1.

Rosettes with fewer than five circles do not generate suitable curvilinear rhombus-shaped slots. We also discard the case of $n = 5$ since, as illustrated in Figure 7, the rhombuses are asymmetric, unlike rosettes with six or more circles. That is, the two innermost sides are shorter than the other two, whereas for $n \geq 6$ all four sides are of equal length. Thus we can specify the number of rings of slots as $\lfloor \frac{n}{2} \rfloor - 2$. For each odd and even case we can also distinguish two types of configurations in which the solutions alternate between being aligned with the y axis or rotated by $\frac{\pi}{n}$. This alternation also coincides with whether the outermost ring contains a four-sided slot or a three-sided one, and depends on $n \bmod 4$.



(a) (b) (c) (d) (e)

Figure 7: Rosettes with 5 to 9 circles

In the rest of this section we only examine the n even, "case (a)" solution. The other cases can be solved in a similar fashion. To find the ellipse in a given slot, the conditions to be satisfied are (see Figure 4) that the ellipse

$f = 0$ meets the circle $g = 0$ tangentially at point (x_g, y_g) and that the ellipse $f = 0$ meets the circle $h = 0$ tangentially at point (x_h, y_h).

We may approach the formulation and solution of these conditions in two ways.

2.2 First approach

In the first approach we consider the normals ∇f, ∇g, and ∇h to each curve. These are orthogonal to the curve tangents, but are not necessarily unit normals, and are merely proportional at the appropriate contact points. Thus, letting subscripts of x and y denote derivatives, we require that at (x_g, y_g):

$$f(x_g, y_g) = 0, \quad g(x_g, y_g) = 0, \quad f_x(x_g, y_g) g_y(x_g, y_g) = f_y(x_g, y_g) g_x(x_g, y_g), \tag{1}$$

while at (x_h, y_h):

$$f(x_h, y_h) = 0, \quad h(x_h, y_h) = 0, \quad f_x(x_h, y_h) h_y(x_h, y_h) = f_y(x_h, y_h) h_x(x_h, y_h). \tag{2}$$

We now proceed by eliminating x_g and y_g from the set of Eqns. 1, and x_h and y_h from the set of Eqns. 2. Either resultant [2] or Gröbner basis [3] algorithms may be used to perform the elimination. The former have the advantage of generally running more quickly, while the latter generally produce less extraneous factors in the solution. In either case, we factorize the solutions obtained and throw away roots which do not correspond to roots of the original problem. The results of the previous step are a pair of equations in A, S and y_0. These are of degree 16 in y_0, 20 in S, and 8 in A, and take up too much space to present here. Finally, we can then in principle eliminate one further variable between these remaining equations, most usefully A, to get a relationship between S and y_0.

2.3 Second approach

The first approach proved intractable even when using *Mathematica*. As an alternative we now rely on the idea that two curves which have tangential contact have second order contact. Suppose a curve $y = F(x)$ is tangent to the x-axis. At this point, it satisfies $y = 0$, and also $dy/dx = 0$, as it meets, and is tangential to, the axis. Analogously, in our problem, we may express the condition that $f(x, y) = 0$ meets $g(x, y) = 0$ with a second order contact at (x_g, y_g) by the following procedure. Firstly

$$\text{eliminate } y \text{ from } f(x, y) = 0 \text{ and } g(x, y) = 0$$

to give

$$s(x) = 0.$$

Then we require both

$$s(x_g) = 0 \quad \text{and} \quad s_x(x_g) = 0.$$

From the last two equations we eliminate the variable, x_g, to produce an equation in A, S and y_0.

Of course, we could just as well eliminate x first rather than y, and take the y derivative instead in the above procedure. Degenerate cases arise for horizontal or vertical contacts between $f = 0$ and $g = 0$, but these do not occur in the problem of interest. The result of carrying out this procedure produces an equation of degree 8 in y_0, degree 4 in S and degree 4 in A (which is omitted here for reasons of space). The equivalent problem is also solved for (x_h, y_h) giving a second equation in A, S and y_0.

Finally, in principle, we can now eliminate one of these remaining variables to again give a single relationship between, say, S and y_0.

3 Practical results

In practice, computer algebra systems are not powerful enough, nor is this paper long enough, to give a general solution! To be able to reach a single equation using either approach we must specialise the problem: we either have to put in numerical values for n and j if we are interested in ellipses of arbitrary shape, *or* we have to choose a special value for S if we want a solution for *any* n and j. We show the results of both specialisations below.

3.1 Fixing n and j

If we fix $n = 6$ and $j = 1$, as a simple case, the second approach of Section 2 above gives the following.

Eliminating x_g and y_g where $f = 0$ and $g = 0$ meet, and similarly, eliminating x_h and y_h where $f = 0$ and $h = 0$ meet gives two equations from which A can be eliminated to produce an explicit polynomial equation of degree 22 in y_0 and 17 in S which nevertheless is still too long to give here. However, if S (or y_0) is chosen, this equation can be solved numerically to find the desired ellipse.

A similar approach also works for other particular numerical values of n and j, although the trigonometric functions of n and j may be more complicated, resulting in lengthier equations.

In practice, for large values of S (greater than about 5, say) the solution is rather ill-conditioned. Unless care is taken the trivial solution $y_0 = S = A = 0$ is found. In addition, the undesirable solution $a = r$ needs to be avoided.

3.2 Fixing $S = 1$

If we try putting in various special values of S, for example 2, or $\frac{1}{2}$, we find that this does not usefully simply matters most of the time. However, the particular case $S = 1$, i.e. where the ellipses in each slot become circles, does lead to a significant simplification. Early in the process, many extra factors produced during elimination, and leading to unwanted solutions, can be removed. The contacts of $f = 0$ with $g = 0$ and with $h = 0$ lead respectively to the two equations

$$\begin{aligned}(4 - A)\ A + 2\ (A - 1)\ {y_0}^2 - {y_0}^4 + 2\,{y_0}^2\ \cos(2\,\theta_g) + 4\,y_0\ \left({y_0}^2 - A\right)\ \sin(\theta_g) &= 0,\\ (4 - A)\ A + 2\ (A - 1)\ {y_0}^2 - {y_0}^4 + 2\,{y_0}^2\ \cos(2\,\theta_h) + 4\,y_0\ \left({y_0}^2 - A\right)\ \sin(\theta_h) &= 0\end{aligned}$$

From these, it turns out that the equation for y_0, the position of the centre of the circle, is linear so can easily be determined.

We can also derive a linear expression for A, and hence a single solution for a, the radius of the circle. We can thus express circular solutions to our problem for *any* n and j

$$a\ (= b) = \frac{2\ (\cos(2\,\theta_g) - \cos(2\,\theta_h))}{6 + \cos(2\,\theta_g) + \cos(\theta_h)^2 + 4\ \sin(\theta_g)\ \sin(\theta_h) - \sin(\theta_h)^2}$$

and

$$y_0 = \frac{8\ (\sin(\theta_g) + \sin(\theta_h))}{6 + \cos(2\,\theta_g) + \cos(2\,\theta_h) + 4\ \sin(\theta_g)\ \sin(\theta_h)}.$$

To solve for a particular case, the values for θ_g and θ_h from Table 1 should be inserted after choosing j and n. This has been done for all values of j for $n = 12$ to produce the pattern shown in Figure 8.

3.3 Maximising the area

As a third experiment we also tried specialising the ellipse in each slot to be the one of maximum area. Starting with the second approach of Section 2, we may perform the final elimination in two alternative ways eliminating either S or A. We arrive at a pair of equations which we will write as

$$E(y_0, S) = 0 \quad \text{and} \quad F(y_0, A) = 0$$

for the family of ellipses. The area of an ellipse is given by $Q = \pi ab = \pi A\sqrt{S}$. This is maximised when $dQ/dA = 0$; note that S depends on A. Thus, the area is maximised when

$$\pi\left(\sqrt{S} + \frac{1}{2\sqrt{S}}\frac{dS}{dA}\right) = 0,$$

or

$$2S + \frac{dS}{dA} = 0$$

However

$$\frac{dE}{dA} = \frac{\partial E}{\partial y_0}\frac{dy_0}{dA} + \frac{\partial E}{\partial S}\frac{dS}{dA} \quad \text{and} \quad \frac{dF}{dA} = \frac{\partial F}{\partial y_0}\frac{dy_0}{dA} + \frac{\partial F}{\partial A}.$$

From these we may obtain

$$\frac{dS}{dA} = \frac{\frac{\partial E}{\partial y_0}\frac{\partial F}{\partial A}}{\frac{\partial F}{\partial y_0}\frac{\partial E}{\partial S}}$$

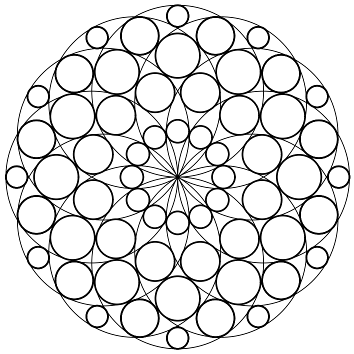


Figure 8: Circles inscribed in the rosette

Figure 9: The Piazza del Campidoglio, Rome

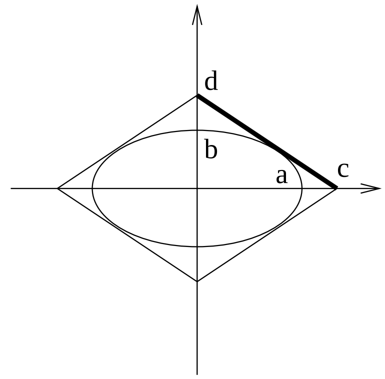


Figure 10: Maximum area ellipse inscribed in a rhombus

and hence the ellipse area is maximised by solving

$$2S\frac{\partial F}{\partial y_0}\frac{\partial E}{\partial S}+\frac{\partial E}{\partial y_0}\frac{\partial F}{\partial A}=0$$

in addition to $E=0$ and $F=0$.

Even in the particular case $n=6$ and $j=1$, this new third equation proved to be of extremely high degree, namely degree 45 in y_0, and degree 17 in both S and A. We return to maximum area ellipses in the following sections.

4 Polygonal circular rosettes

Another possible simplification is to approximate the circular arcs making up the sides of each slot by straight lines joining the original vertex positions. In fact, such a polygonal rosette was designed by Michelangelo for the Piazza del Campidoglio at the top of Capitoline Hill (see Figure 9).[3] Since each rhombus is symmetrical about both axes (unlike the previous case with circular arcs) we need just consider one of its edges to find the maximum area inscribed ellipse. Taking the first quadrant, as shown in Figure 10, suppose the equation of the bold edge is

$$y=-\frac{d}{c}(x-c).$$

Combining the constraints that the ellipse and line have equal tangents and touch, and eliminating x and y we obtain

$$c^2\left(\frac{d}{b^2}-\frac{1}{d}\right)=\frac{a^2\,d}{b^2}\quad\Rightarrow\quad a=\frac{c\sqrt{-b^2+d^2}}{d}.$$

The maximum area ellipse is found when

$$dQ=\pi\left(a\;db+b\;da\right)=0,$$

and so the condition is

$$\frac{c\sqrt{-b^2+d^2}}{d}-\frac{c\,b^2}{d\sqrt{-b^2+d^2}}=0$$

which yields

$$b=\frac{d}{\sqrt{2}},\quad a=\frac{c}{\sqrt{2}}.$$

It is also straightforward to show that the ellipse makes contact with the midpoint of the line. Consequently, the ellipses on either side of the lines are also tangent to each other, as shown in Figure 11.

[3]There were even longer delays involved in this project, and the pavement was not laid until 1940!

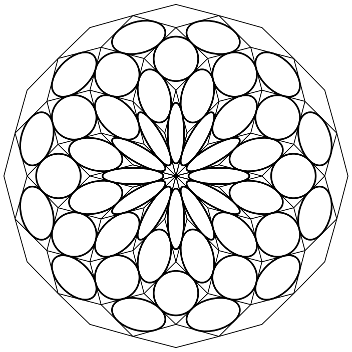


Figure 11: Maximum area ellipses inscribed in the polygonal approximation of the rosette

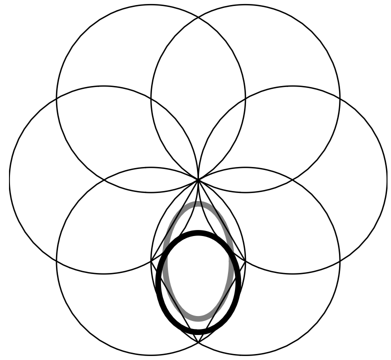


Figure 12: The maximum area inscribed ellipses for both a circular and polygonal rosette.

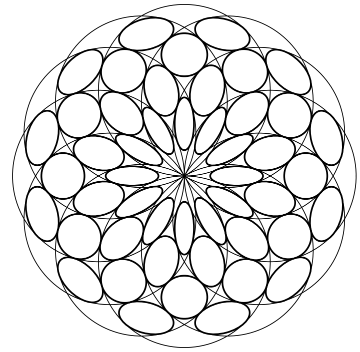


Figure 13: Maximum area ellipses inscribed in the circular rosette

5 Comparing slots in circular and polygonal rosettes

Here we compare the curvilinear rhombus formed by the circular and the regular straight-sided one from the polygonal rosette. In several ways both rhombuses are similar. They have the same area, and in both cases are formed by four equal length lines or arcs. Does the change in shape affect the size and shape of the inscribed ellipse?

For the simple case of an inscribed circle at $n = 6$ and $j = 1$ we can easily analytically determine the radius for the curvilinear case ($r_c = \frac{6}{13}$) and for the polygonal rosette ($r_p = \frac{\sqrt{3}}{4}$). Their ratio is $\frac{r_c}{r_p} \approx 1.06588$, and so the curvilinear slots admits a 7% larger circle.

For the more general case we find (numerically) that the maximum area inscribed ellipses in the circular rosette are still larger than the polygonal rosette, but only by smaller amounts (e.g. about 4% for $n = 6$ as shown in figure 12, and 1% for $n = 12$). As n increases the differences between the inscribed ellipses in the circular and polygonal rosettes decrease.

6 A conjecture

The circular rosette with maximum area inscribed ellipses (determined numerically from the equations in Section 3.3) is shown in Figure 13 for $n = 12$. Notice that the ellipses appear to be in contact with their neighbours, although we have been unable to prove this to be the case. We conjecture that the ellipses are indeed in contact, and challenge readers to prove or disprove this.

7 Postscript

(a) (b) (c)

Figure 14: The patterns from (a) the ceiling and (b) the central floor aisle. (c) Michelangelo's stairway leading to the reading room.

The rosette was not the only application of the ellipse within the library's design. It also appears in the ceiling and central floor designs shown in Figure 14a&b. A more prominent example is in the staircase in the entrance hall, shown in Figure 14c, and built in Michelangelo's absence by Ammannati in 1559, following a clay model. It consists of three flights of steps; the outer ones are quadrilaterals, the central ones are bounded by convex elliptical arcs, and the bottom three steps are complete ellipses. This unusual design is highly thought of, and has been variously described as "an explosion of originality", "a wavelike crescendo", "spilling from the library door", etc.

In fact, although the Classical world preferred the circle, from the Baroque period onwards the dynamism of the ellipse has been highly rated. For instance, the Victorian architect and designer Owen Jones [6] stated that: "In the best periods of art, all mouldings and ornaments were founded on curves of the higher order, such as the conic section; whilst, when art declined, circles and compasswork were much more dominant." Similarly, at around the same time the designer George Phillips [10] wrote that "Lines of varied or curvilinear character are essentially those of beauty." And compared to circles, "An oval . . . has a much greater degree of lightness".

In this light the Medici panel with its rotated circles and radiating ellipses can be seen as a harmonious combination of the Classical and Mannerist sensibilities. Moreover, in addition to its aesthetic charms we have found that this hidden panel has hidden geometric complexities and attractions.

8 Acknowledgements

We would like to thank Ben Nicholson and James Ackerman for helpful discussions and material.

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