A new measure for the rectilinearity of polygons

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ABSTRACT

A polygon P is said to be rectilinear if all interior angles of P belong to the set $\{\pi/2, 3\pi/2\}$. In this paper we establish the mapping

$$\mathcal{R}(P) = \frac{\pi}{\pi - 2 \cdot \sqrt{2}} \cdot \left(\max_{\alpha \in [0, 2\pi]} \frac{\mathcal{P}_1(P, \alpha)}{\sqrt{2} \cdot \mathcal{P}_2(P)} - \frac{2 \cdot \sqrt{2}}{\pi} \right)$$

where P is an arbitrary polygon, $\mathcal{P}_2(P)$ denotes the Euclidean perimeter of P, while $\mathcal{P}_1(P, \alpha)$ is the perimeter in the sense of l_1 metrics of the polygon obtained by the rotation of P by angle α with the origin as the center of the applied rotation. It turns out that $\mathcal{R}(P)$ can be used as an estimate for the rectilinearity of P. Precisely, $\mathcal{R}(P)$ has the following desirable properties:

- any polygon P has the estimated rectilinearity $\mathcal{R}(P)$ which is a number from $[0, 1]$;
- $\mathcal{R}(P)=1$ if and only if P is a rectilinear polygon;
- $\inf_{P \in \Pi} \mathcal{R}(P) = 0$, where Π denotes the set of all polygons;
- a polygon's rectilinearity measure is invariant under similarity transformations.

The proposed rectilinearity measure can be an alternative for the recently described measure $\mathcal{R}_1(P)$.[1] Those rectilinearity measures are essentially different since there is no monotonic function f, such that $f(\mathcal{R}_1(P)) = \mathcal{R}(P)$, that holds for all $P \in \Pi$.

A simple procedure for computing $\mathcal{R}(P)$ for a given polygon P is described as well.

Keywords: Shape, polygons, rectilinearity, measurement.

1. INTRODUCTION

Shape plays an important part in the processing of visual information, and is actively being investigated in a wide spectrum of areas, from art[2] through to science.[3] Within computer vision there have been many applications of shape to aid in the analysis of images, and standard shape descriptors include compactness, eccentricity,[4] circularity,[5] ellipticity,[6] and rectangularity.[7]

This paper describes a shape measure that has received little attention: *rectilinearity*. While there exist a variety of approaches to computing the related measure of *rectangularity*,[7] rectilinearity covers a wider space of shapes since the number of sides of the model shape is variable. Here we define a rectilinearity measure which can be used as an alternative to that recently described in Ref. 1.

One of possible applications of the obtained results is to provide a useful tool for the analysis of buildings in aerial photographs, since many buildings appear rectilinear from an overhead view. Over the last 10-20 years there has been considerable research in this area with the goal of providing automatic photointerpretation which would be particularly useful for cartographers.[8–11]

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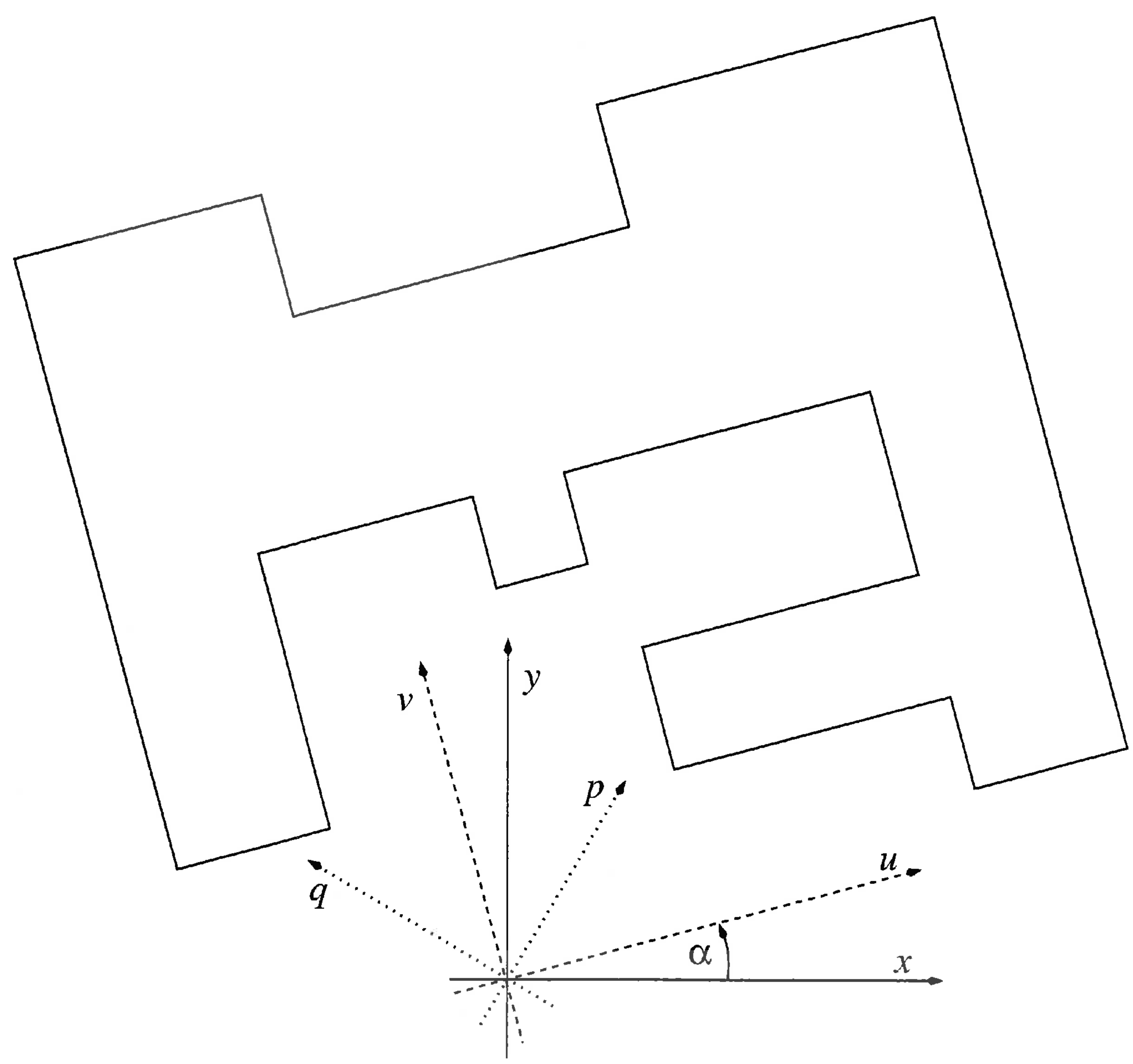

Figure 1. For the given rectilinear 20-gon P, its l_1 perimeter $\mathcal{P}_1(P)$ has the minimum value if the coordinate axes are chosen to be parallel with u and v, while it reaches its maximum if the coordinate axes are parallel to p and q. The minimum and maximum values correspond to $\mathcal{P}_1(P, 2\pi - \alpha)$ and $\mathcal{P}_1(P, \frac{7\pi}{4} - \alpha)$, respectively, if x and y are taken to be the coordinate axes.

2. DEFINITIONS AND DENOTATIONS

We start this section with the formal definition of rectilinear polygons (see Fig. 1 for an example).

DEFINITION 2.1. *A polygon P is rectilinear if its interior angles belong to the set* $\left\{\frac{\pi}{2}, \frac{3\cdot\pi}{2}\right\}$.

Also, we will use the following definitions and denotations (see Fig. 2 and Fig. 3 for some illustrations). The set of all polygons will be denoted by Π. For a given n-gon P having vertices denoted by $A_0, A_1, \ldots, A_{n-1}, A_n = A_0$, its edges will be denoted $e_i = [A_{i-1}, A_i]$ for $i = 1, 2, \ldots, n$. The Euclidean length of the straight line segment $e = [(x_1, y_1), (x_2, y_2)]$ is $l_2(e) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, while the length of e according to the l_1 metric is $l_1(e) = |x_1 - x_2| + |y_1 - y_2|$.

$\mathcal{P}_2(P)$ will denote the Euclidean perimeter of P, while $\mathcal{P}_1(P)$ will denote the perimeter of P in the sense of l_1 metrics. So,

$$\mathcal{P}_2(P) = \sum_{e_i \text{ is an edge of } P} l_2(e_i) \quad \text{and} \quad \mathcal{P}_1(P) = \sum_{e_i \text{ is an edge of } P} l_1(e_i).$$

Since isometric polygons do not necessarily have the same perimeter under the l_1 metric, we shall use $\mathcal{P}_1(P, \alpha)$ for the l_1 perimeter of the polygon which is obtained by the rotating P by the angle α with the origin as the centre of rotation. If the same rotation is applied to the edge e, the l_1 perimeter of the obtained edge will be denoted as $l_1(e, \alpha)$.

If the oriented angle between the positively oriented x-axis and the vector $\overrightarrow{A_{i-1}A_i}$ is denoted by ϕ_i ($i = 1, 2, \ldots n$), then obviously $l_1(e_i, \alpha) = l_2(e_i) \cdot (|\cos(\phi_i + \alpha)| + |\sin(\phi_i + \alpha)|)$. Thus, by using $1 \leq |\cos\beta| + |\sin\beta| \leq \sqrt{2}$

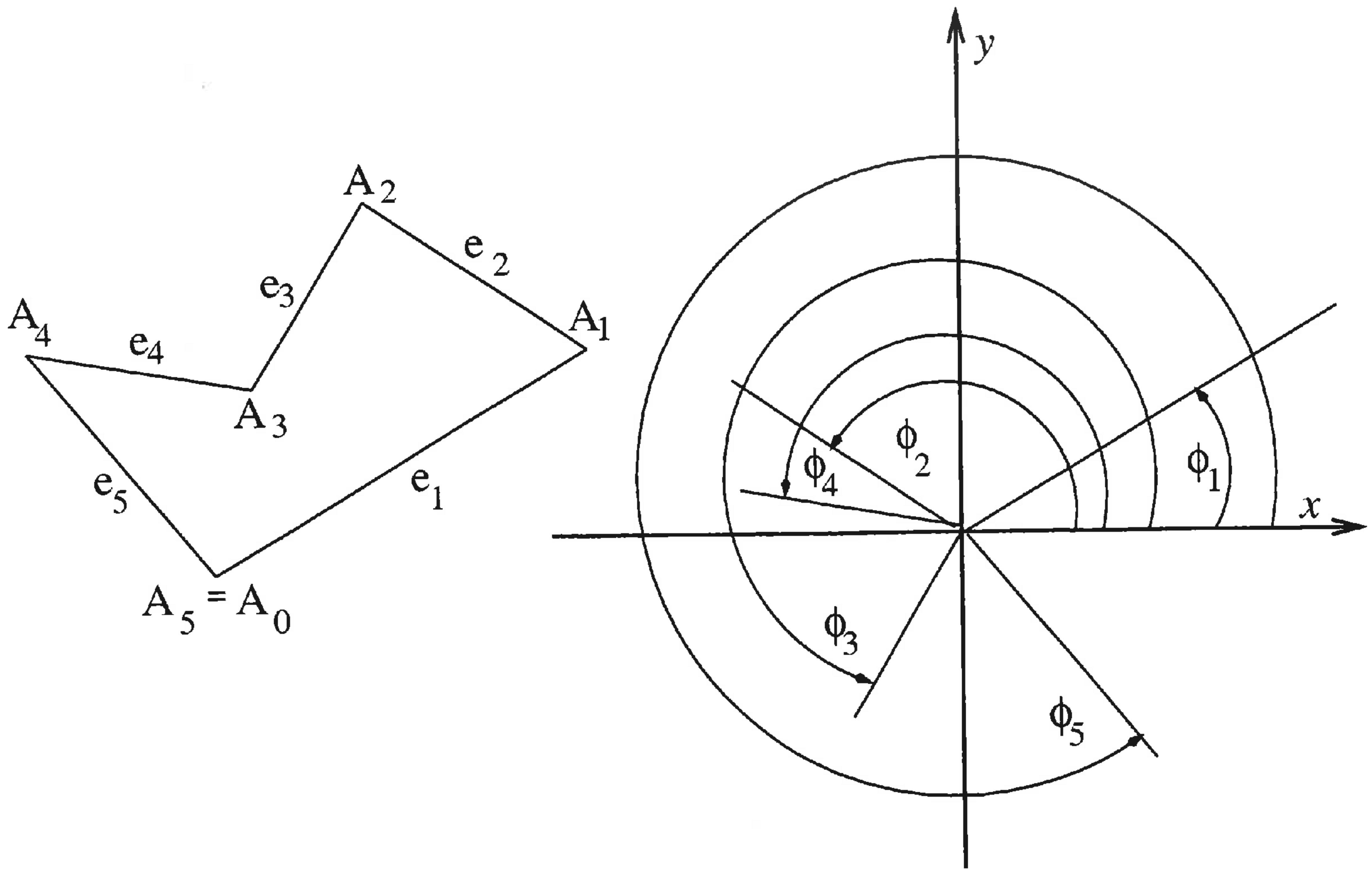

Figure 2. A non rectilinear 5-gon. The used denotations are illustrated.

(for any β), we have the next two relations between $\mathcal{P}_1(P)$ and $\mathcal{P}_2(P)$.
The first one is

$$\mathcal{P}_2(P) = \sum_{i=1}^{n} l_2(e_i) \leq \sum_{i=1}^{n} l_2(e_i) \cdot (|\cos(\phi_i + \alpha)| + |\sin(\phi_i + \alpha)|) = \mathcal{P}_1(P, \alpha) \tag{1}$$

NOTE 2.1. *Let us notice that* $\mathcal{P}_2(P) = \mathcal{P}_1(P, \alpha)$ *if and only if* $|\cos(\phi_i + \alpha)| + |\sin(\phi_i + \alpha)| = 1$ *holds for all edges* e_i, $1 \leq i \leq n$. The second one is

$$\mathcal{P}_1(P, \alpha) = \sum_{i=1}^{n} l_2(e_i) \cdot (|\cos(\phi_i + \alpha)| + |\sin(\phi_i + \alpha)|) \leq \sqrt{2} \cdot \sum_{i=1}^{n} l_2(e_i) \leq \sqrt{2} \cdot \mathcal{P}_2(P). \tag{2}$$

NOTE 2.2. *Similarly to Note 2.1,* $\mathcal{P}_1(P) = \sqrt{2} \cdot \mathcal{P}_1(P, \alpha)$ *if and only if* $|\cos(\phi_i + \alpha)| + |\sin(\phi_i + \alpha)| = \sqrt{2}$ *holds for all edges* e_i, $1 \leq i \leq n$.

We will exploit the following property of rectilinear polygons which is derived from (2) and formulated as a theorem.

THEOREM 2.2. *A given polygon P is rectilinear if and only if there exists* α *such that*

$$\mathcal{P}_1(P, \alpha) = \sqrt{2} \cdot \mathcal{P}_2(P).$$

Proof. If the given polygon P is rectilinear, then the rotation of P, such that the angles between edges of P and the coordinate axes belong to the set $\left\{\frac{\pi}{4}, \frac{3 \cdot \pi}{4}, \frac{5 \cdot \pi}{4}, \frac{7 \cdot \pi}{4}\right\}$, preserves the equation $\mathcal{P}_1(P, \alpha) = \sqrt{2} \cdot \mathcal{P}_2(P)$ where α is the rotation angle.

On the other hand, if $\mathcal{P}_1(P, \alpha) = \sqrt{2} \cdot \mathcal{P}_2(P)$ then (by Note 2.2) it must be $|\cos(\phi_i + \alpha)| + |\sin(\phi_i + \alpha)| = \sqrt{2}$ for all edges e_i, $1 \leq i \leq n$, of the given n-gon P. That implies $|\cos(\phi_i + \alpha)| = |\sin(\phi_i + \alpha)| = \frac{\sqrt{2}}{2}$ for all i, with $1 \leq i \leq n$ – but it means that all edges of P are either parallel or orthogonal to the same line. This completes the proof. □

3. THE BASIC IDEA AND NECESSARY MATHEMATICS

Theorem 2.2 gives a useful characterisation of rectilinear polygons and gives the basic idea for the polygon rectilinearity measurement described in this paper. In the first stage, Theorem 2.2 together with $\mathcal{P}_1(P) \leq \sqrt{2}\cdot\mathcal{P}_2(P)$ (see (2)) suggests that the ratio $\dfrac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ can be used as a rectilinearity measure for the polygon P. More precisely, the ratio $\dfrac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ has the following "good" properties:

- a1) it is a positive number;
- a2) it is defined for any polygon P;
- a3) it can be calculated easily;
- a4) for any non rectilinear polygon it is strictly less than 1. For any given rectilinear polygon it is exactly 1 if the coordinate axes are suitably chosen.

But, on the other hand $\dfrac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ has the following "bad" properties:

- b1) it is not invariant under similarity (even isometric) transformations;
- b2) the infimum for the set of values of $\mathcal{Q}(P,\alpha) = \dfrac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ is not zero. For an example, it can be seen easily (from (1)) that there exists no polygon P such that $\dfrac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)} \in \left(0, \dfrac{\sqrt{2}}{2}\right)$.

In this section we develop necessary mathematical tools in order to define a function $\mathcal{R}(P)$ which satisfies a1)–a4) but not b1) and b2).

The problem described by b1) can be avoided by considering $\max\limits_{\alpha\in[0,2\pi]} \dfrac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ instead of $\dfrac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$, but it opens the question of how to compute this maximum.

Further, (1) and (2) give

$$\frac{\sqrt{2}}{2} \leq \frac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)} \leq 1\,, \quad \text{and consequently,} \quad \frac{\sqrt{2}}{2} \leq \max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)} \leq 1$$

for any polygon P. But, while the inequality $\dfrac{\mathcal{P}_1(P)}{\sqrt{2}\cdot\mathcal{P}_2(P)} \leq 1$ is sharp, and moreover, $\max\limits_{\alpha\in[0,2\pi]} \dfrac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_1(P)} = 1$ is satisfied if and only if P is a rectilinear polygon (due to Theorem 2.2), it can be seen easily that there exists no polygon P such that $\max\limits_{\alpha\in[0,2\pi]} \dfrac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)} = \dfrac{\sqrt{2}}{2}$. Namely, if an n-gon P satisfies the last equality, then for some α_0 we have $\dfrac{\mathcal{P}_1(P,\alpha_0)}{\sqrt{2}\cdot\mathcal{P}_1(P)} = \dfrac{\sqrt{2}}{2}$ which (see Note 2.1) would imply $l_1(e_i,\alpha_0) = l_2(e_i)$ or, equivalently, $\phi_i + \alpha_0 \in \{0, \pi/2, 3\pi/2, 2\pi\}$ for any edge e_i where $1 \leq i \leq n$. So, P must be rectilinear and due to Theorem 2.2, the considered maximum must be equal to 1, which is a contradiction.

So, for our purpose it is necessary to determine the maximal possible μ such that the $\max\limits_{\alpha\in[0,2\pi]} \dfrac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ belongs to the interval $[\mu, 1]$ for any polygon P. The next two lemmas together show $\mu = \dfrac{2\cdot\sqrt{2}}{\pi}$.

LEMMA 3.1. *The inequality*

$$\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_1(P)} > \frac{2\cdot\sqrt{2}}{\pi}$$

holds for any polygon P.

Proof. We prove the statement by a contradiction. Let us assume the contrary, i.e., there exists an n-gon P such that

$$\frac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)} < \frac{\sqrt{2}\cdot 2}{\pi} \qquad \text{or equivalently,} \qquad \frac{\mathcal{P}_1(P,\alpha)}{\mathcal{P}_2(P)} < \frac{4}{\pi}, \qquad \text{for any} \quad \alpha\in[0,2\pi].$$

Since $\frac{\mathcal{P}_1(P,\alpha)}{\mathcal{P}_2(P)}$ is a continuous, nonconstant function which is less than or equal to $\frac{4}{\pi}$, we have

$$\int_0^{2\pi} \frac{\mathcal{P}_1(P,\alpha)}{\mathcal{P}_2(P)}\cdot d\alpha \;<\; \int_0^{2\pi} \frac{4}{\pi}\cdot d\alpha \;=\; 8\,. \tag{3}$$

By using (3) we have:

$$8 \;>\; \int_0^{2\pi} \frac{\mathcal{P}_1(P,\alpha)}{\mathcal{P}_2(P)}\cdot d\alpha \;=\; \frac{1}{\mathcal{P}_2(P)}\cdot\int_0^{2\pi}\left(\sum_{i=1}^{n} l_1(e_i,\alpha)\right)\cdot d\alpha \;=\; \frac{1}{\mathcal{P}_2(P)}\cdot\sum_{i=1}^{n}\left(\int_0^{2\pi} l_1(e_i,\alpha)\cdot d\alpha\right)$$

$$=\frac{1}{\mathcal{P}_2(P)}\cdot\sum_{i=1}^{n}\left(\int_0^{2\pi} l_2(e_i)\cdot(|\sin(\phi_i+\alpha)|+|\cos(\phi_i+\alpha)|)\cdot d\alpha\right) \;=\; \frac{1}{\mathcal{P}_2(P)}\cdot\left(\sum_{i=1}^{n} 8\cdot l_2(e_i)\right) \;=\; 8.$$

The contradiction $8 > 8$ finishes the proof. □

So, in accordance with the above discussion, Lemma 3.1 shows that the required number μ is not smaller than $\frac{2\cdot\sqrt{2}}{\pi}$. The next lemma shows that μ is not bigger than $\frac{2\cdot\sqrt{2}}{\pi}$ and consequently $\mu = \frac{2\cdot\sqrt{2}}{\pi}$.

LEMMA 3.2. *The infimum for the set of values* $\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ *for all possible choices of polygon P is* $\frac{2\cdot\sqrt{2}}{\pi}$, *i.e.,*

$$\inf_{P\in\Pi}\left\{\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}\right\} \;=\; \frac{2\cdot\sqrt{2}}{\pi}$$

Proof. To prove the statement (taking into acaunt Lemma 3.1) it is enough to find a sequence of polygons $P_3, P_4, P_5, \ldots$ such that

$$\lim_{n\to\infty}\left(\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P_n,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P_n)}\right) \;=\; \frac{2\cdot\sqrt{2}}{\pi}\,.$$

We will prove that the sequence of regular n-gons P_n inscribed into the unit circle satisfies the previous equality. Namely, it can be easily seen that the sequence of the Euclidean perimeters of P_n tends to the perimeter of the unit circle, i.e.,

$$\lim_{n\to\infty} \mathcal{P}_2(P_n) \;=\; 2\cdot\pi \tag{4}$$

but also

$$\lim_{n\to\infty} \mathcal{P}_1(P_n,\alpha) \;=\; 8 \tag{5}$$

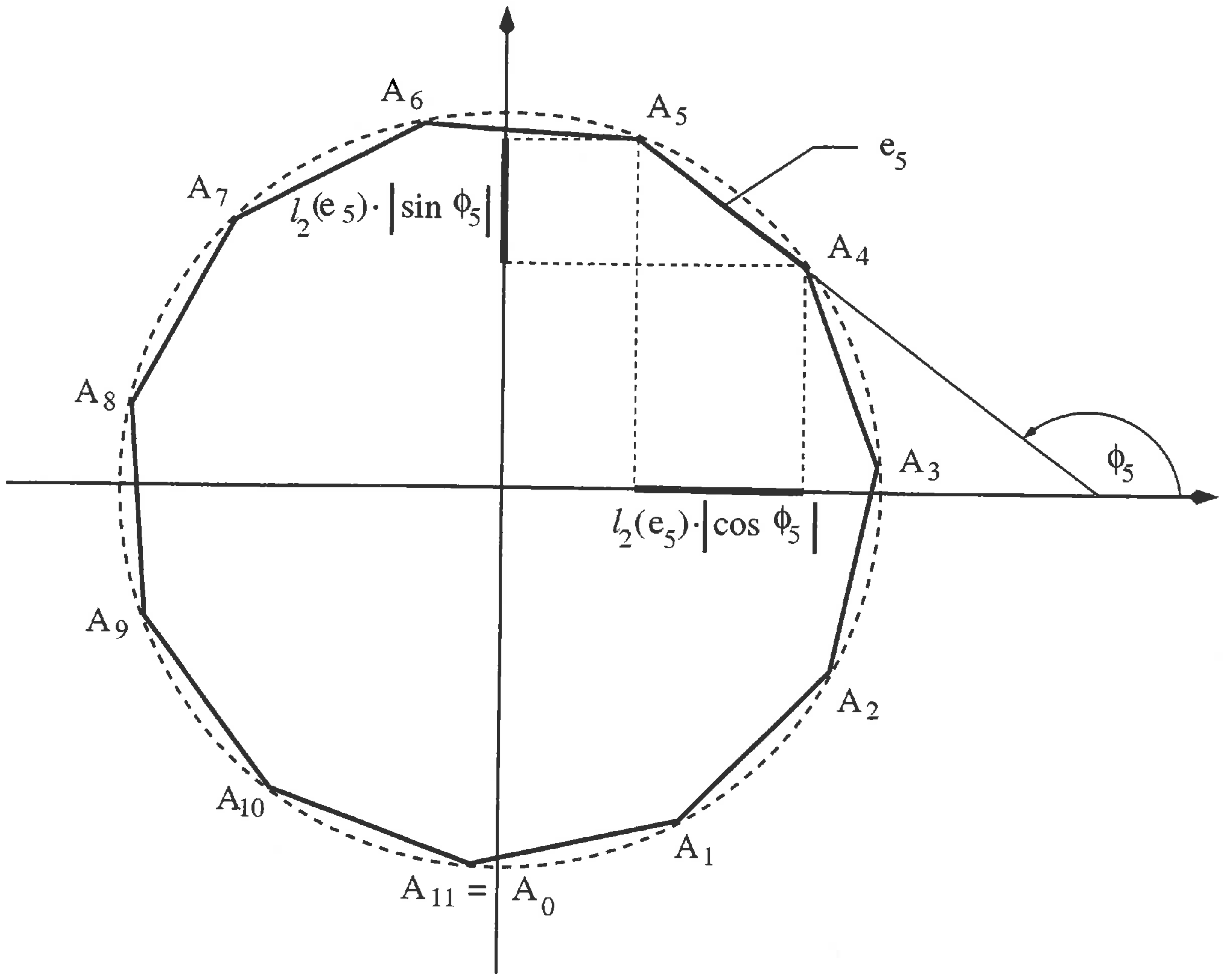

Figure 3. A regular 11-gon inscribed into the unit circle with the centre placed at the origin.

holds independently on the choice of α. Precisely, if it is considered that (for any $\alpha \in [0, 2\pi]$) the l_1 perimeter $\mathcal{P}_1(P_n, \alpha)$ equals the sum of the projections of all edges of P_n onto x and y axes, than it follows that this sum tends to 8 as $n \to \infty$ (see Fig. 3 for an illustration). Since the limits in (4) and (5) are independent on α we have

$$\lim_{n\to\infty} \frac{\mathcal{P}_1(P_n, \alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P_n)} = \lim_{n\to\infty}\left(\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P_n, \alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P_n)}\right) = \frac{2\cdot\sqrt{2}}{\pi},$$

which finishes the proof. □

4. A RECTILINEARITY MEASURE

By using the proposed properties of the function $\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P, \alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ we give the following definition for a rectilinearity measurement of polygons.

DEFINITION 4.1. *For arbitrary polygon P we define its rectilinearity $\mathcal{R}(P)$ as:*

$$\mathcal{R}(P) = \frac{\pi}{\pi - 2\cdot\sqrt{2}}\cdot\left(\max_{\alpha\in[0,2\pi]} \frac{\mathcal{P}_1(P, \alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)} - \frac{2\cdot\sqrt{2}}{\pi}\right).$$

The following theorem summarises the properties of the polygon rectilinearity measure proposed here.

THEOREM 4.2. *For any polygon P, we have:*

i) $\mathcal{R}(P)$ *is determined and* $\mathcal{R}(P) \in (0, 1]$;

ii) $\mathcal{R}(P) = 1$ *if and only if* P *is a rectilinear polygon;*

iii) $\inf_{P\in\Pi} (\mathcal{R}(P)) = 0$;

iv) $\mathcal{R}(P)$ is invariant under similarity transformations.

Proof. The item *i)* follows from (2), definition of $\mathcal{R}(P)$, and Lemma 3.1. The item *ii)* is a direct consequence of Theorem 2.2. The item *iii)* is the statement of Lemma 3.2. To prove *iv)* let us notice that $\mathcal{R}(P)$ is invariant under all isometric transformations – whch follows from the definition. Also, $\dfrac{\mathcal{P}_1(P,\alpha)}{\sqrt{2}\cdot\mathcal{P}_2(P)}$ and consequently $\mathcal{R}(P)$ are invariants under any transformation of the form $(x,y)\rightarrow(\lambda\cdot x,\lambda\cdot y)$ for any choice of $\lambda\neq 0$, P, and α. That completes the proof. □

Some examples of polygons with their computed rectilinearity are given in Fig. 4.

5. COMPUTATION OF $\mathcal{R}(P)$

The question which remains open is how to compute $\mathcal{R}(P)$ effectively for a given polygon P. Since $\mathcal{P}_2(P)$ can be easily calculated from the vertices of P it remains to describe the computation of the maximum value of $\mathcal{P}_1(P,\alpha)$ when α varies from 0 to 2π. In this section we describe a procedure for such a computation.

Let us consider an edge e_i $(1\leq i\leq n)$ of a given n-gon P. From trivial equalities

$$l_1(e_i,\alpha)=\begin{cases} l_2(e_i)\cdot\cos(\phi_i+\alpha)+l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{for} \quad \alpha\in[-\phi_i,\frac{\pi}{2}-\phi_i] \\ -l_2(e_i)\cdot\cos(\phi_i+\alpha)+l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{for} \quad \alpha\in[\frac{\pi}{2}-\phi_i,\pi-\phi_i] \\ -l_2(e_i)\cdot\cos(\phi_i+\alpha)-l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{for} \quad \alpha\in[\pi-\phi_i,\frac{3\pi}{2}-\phi_i] \\ l_2(e_i)\cdot\cos(\phi_i+\alpha)-l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{for} \quad \alpha\in[\frac{3\pi}{2}-\phi_i,2\pi-\phi_i]\,. \end{cases}$$

it holds that there is an integer $k\leq 4\cdot n$ and a sequence $0\leq\alpha_1<\alpha_2<\ldots<\alpha_k\leq 2\pi$ such that

$$\mathcal{P}_1(P,\alpha)=\begin{cases} \sum\limits_{i=1}^{n} a_{1,i}\cdot l_2(e_i)\cdot\cos(\phi_i+\alpha)+b_{1,i}\cdot l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{if } \alpha\in[\alpha_1,\alpha_2] \\ \sum\limits_{i=1}^{n} a_{2,i}\cdot l_2(e_i)\cdot\cos(\phi_i+\alpha)+b_{2,i}\cdot l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{if } \alpha\in[\alpha_2,\alpha_3] \\ \cdots\cdots\cdots & \\ \sum\limits_{i=1}^{n} a_{k-1,i}\cdot l_2(e_i)\cdot\cos(\phi_i+\alpha)+b_{k-1,i}\cdot l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{if } \alpha\in[\alpha_{k-1},\alpha_k] \\ \sum\limits_{i=1}^{n} a_{k,i}\cdot l_2(e_i)\cdot\cos(\phi_i+\alpha)+b_{k,i}\cdot l_2(e_i)\cdot\sin(\phi_i+\alpha) & \text{if } \alpha\in[\alpha_k,2\pi+\alpha_1]\,, \end{cases}$$

where

$$\{a_{j,i},b_{j,i}\mid 1\leq j\leq k,\ 1\leq i\leq n\}\ \subset\ \{+1,-1\},$$

or precisely, for any $i\in\{1,2,\ldots,n\}$ and any $j\in\{1,2,\ldots,k\}$

$$a_{j,i} = 1 \quad \text{if} \quad \cos(\phi_i+\alpha)>0 \quad \text{for} \quad \alpha\in(\alpha_j,\alpha_{j+1}) \tag{6}$$
$$a_{j,i} = -1 \quad \text{if} \quad \cos(\phi_i+\alpha)<0 \quad \text{for} \quad \alpha\in(\alpha_j,\alpha_{j+1}) \tag{7}$$

and analogously,

$$b_{j,i} = 1 \quad \text{if} \quad \sin(\phi_i+\alpha)>0 \quad \text{for} \quad \alpha\in(\alpha_j,\alpha_{j+1}) \tag{8}$$
$$b_{j,i} = -1 \quad \text{if} \quad \sin(\phi_i+\alpha)<0 \quad \text{for} \quad \alpha\in(\alpha_j,\alpha_{j+1}) \tag{9}$$

NOTE 5.1. *For any angle $\alpha_p\in\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subset[0,\ 2\pi]$ there is an edge e_q, $1\leq q\leq n$ such that the rotation of e_q for the angle α_p belong to one of coordinate axes. Since some of such angles can coincide, depending of the given n-gon the inequality $k\leq 4\cdot n$ can be strict.*

What is important for us is

$$0<\mathcal{P}_1(P,\alpha)=-\mathcal{P}_1''(P,\alpha) \quad \text{if} \quad \alpha\notin\{\alpha_1,\alpha_2,\ldots,\alpha_k\}. \tag{10}$$

By solving the differential equation from (10), or by applying the formulas

$$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

$$\sin(\alpha+\beta) = \cos\alpha\cdot\sin\beta + \sin\alpha\cdot\cos\beta$$

to the previously given explicit expression for $\mathcal{P}_1(P,\alpha)$, we have:

$$\mathcal{P}_1(P,\alpha) = \begin{cases} c_1\cdot\cos\alpha + d_1\cdot\sin\alpha & \text{for} \quad \alpha\in[\alpha_1,\alpha_2] \\ c_2\cdot\cos\alpha + d_2\cdot\sin\alpha & \text{for} \quad \alpha\in[\alpha_2,\alpha_3] \\ \dots\dots\dots & \\ c_{k-1}\cdot\cos\alpha + d_{k-1}\cdot\sin\alpha & \text{for} \quad \alpha\in[\alpha_{k-1},\alpha_k] \\ c_k\cdot\cos\alpha + d_k\cdot\sin\alpha & \text{for} \quad \alpha\in[\alpha_k, 2\pi+\alpha_1]\,, \end{cases}$$

where the constants $c_1, d_1, c_2, d_2, \dots, c_k, d_k$ can be calculated by solving the systems

$$\mathcal{P}_1(P,\alpha_j) = c_j\cdot\cos\alpha_j + d_j\cdot\sin\alpha_j \tag{11}$$

$$\mathcal{P}_1(P,\alpha_{j+1}) = c_j\cdot\cos\alpha_{j+1} + d_j\cdot\sin\alpha_{j+1} \tag{12}$$

for any $j\in\{1,2,\dots,k\}$, or directly as

$$c_j = \sum_{i=1}^{n}(a_{j,i}\cdot l_2(e_i)\cdot\cos\phi_i + b_{i,j}\cdot l_2(e_i)\cdot\sin\phi_i) \qquad \text{for} \quad j=1,2,\dots,k \tag{13}$$

and

$$d_j = \sum_{i=1}^{n}(-a_{j,i}\cdot l_2(e_i)\cdot\sin\phi_i + b_{i,j}\cdot l_2(e_i)\cdot\cos\phi_i) \qquad \text{for} \quad j=1,2,\dots,k\,. \tag{14}$$

For a given j $(1\le j\le k)$, if

$$\mathcal{P}_1'(P,\alpha) = -c_j\cdot\sin\alpha + d_j\cdot\cos\alpha = 0 \tag{15}$$

has a solution β_j which belongs to the open interval (α_j,α_{j+1}), this solution is unique on the given interval (because of (10)). Further, $\mathcal{P}_1''(P,\beta_j)<0$ gives

$$\mathcal{P}_1(P,\beta_j) = \max_{\alpha\in(\alpha_j,\alpha_{j+1})}\mathcal{P}_1(P,\alpha).$$

If such a solution does not exist then $\mathcal{P}_1(P,\alpha)$ reaches the maximum in one of the interval endpoints, i.e.,

$$\max_{\alpha\in(\alpha_j,\alpha_{j+1})}\mathcal{P}_1(P,\alpha) = \max\{\mathcal{P}_1(P,\alpha_j),\ \mathcal{P}_1(P,\alpha_{j+1})\}. \tag{16}$$

In any case,

$$\Gamma_j = \max_{\alpha\in(\alpha_j,\alpha_{j+1})}\mathcal{P}_1(P,\alpha)$$

can be calculated by comparing at most three values: $\mathcal{P}_1(P,\alpha_j)$, $\mathcal{P}_1(P,\alpha_{j+1})$, and $\mathcal{P}_1(P,\beta_j)$, if $\beta_j\in(\alpha_j,\alpha_{j+1})$ satisfying (15) exists. Finally, for a given polygon P the function $\mathcal{P}_1(P,\alpha)$ reaches its maximum on $[0,2\pi]$ and this maximum is:

$$\max_{\alpha\in[0,2\pi]}\mathcal{P}_1(P,\alpha) = \max_{1\le j\le k}\{\Gamma_j\}.$$

So, we have the next simple procedure for the computing of $\mathcal{R}(P)$.

PROCEDURE $\mathcal{R}(P)$ *Computation*

Input: The vertices $A_0, A_1, \dots, A_{n-1}$ of a given n-gon P.

1. **Step.**
For any i, $i = 1, 2, \ldots n$, compute the angle ϕ_i, and compute the angle-values $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ in accordance with Note 5.1.
Sort them in the increasing order, i.e., $0 \leq \alpha_1 \leq \alpha_2 < \ldots < \alpha_k \leq 2\pi$.

2. **Step.**
Assign either $+1$ or -1 to $a_{j,i}$ and $b_{j,i}$ for any i from $\{1, 2, \ldots, n\}$ and any j from $\{1, 2, \ldots, k\}$
($\star$ in accordance with (6)-(9) $\star$).

3. **Step.**
Compute c_j and d_j by using (11) and (12) (or equivalently (13) and (14)).

4. **Step.**
For any $j \in \{1, 2, \ldots, k\}$
If the equation (15) has the solution β_j on the interval (α_j, α_{j+1})
then compute Γ_j as $\mathcal{P}_1(P, \beta_j)$
else compute Γ_j as $\max\{\mathcal{P}_1(P, \alpha_j),\ \mathcal{P}_1(P, \alpha_{j+1})\}$
($\star$ in accordance with (16). Note: $\alpha_{k+1} = \alpha_1 + 2\pi$ $\star$).

5. **Step.**
Determine Γ as the maximum of $\{\Gamma_1, \Gamma_2, \ldots \Gamma_k\}$ then compute $\mathcal{R}(P)$ as

$$\mathcal{R}(P) = \frac{\pi}{\pi - 2 \cdot \sqrt{2}} \cdot \left(\frac{\Gamma}{\sqrt{2} \cdot \mathcal{P}_2(P)} - \frac{2 \cdot \sqrt{2}}{\pi} \right)$$

($\star$ in accordance with Definition 4.1 $\star$).

Output: $\mathcal{R}(P)$.

6. SOME EXAMPLES AND CONCLUDING REMARKS

The rectilinearity measure is applied to a (perfect) rectilinear polygon in the top left hand polygon in Fig. 4 which is then degraded in various ways. The first row demonstrates the effect of increasing levels of local noise applied to the polygon's vertices. In the second row the polygon is edited, eliminating vertices, which effectively rounds corners and increases its convexity. A shearing transformation is applied in the third row. Finally, the polygon is warped, and the axis aligned edges are increasingly rounded. All examples show that the rectilinearity measure is well behaved; increasing distortion consistently decreases the computed value. Note also that the orientations that maximised $\mathcal{Q}(P, \alpha)$ match our expectations except at high noise levels when the rectilinearity measure has dropped close to zero. For each of the maximally degraded polygons (i.e. the rightmost examples in each row) Fig. 5 plots $\mathcal{Q}(P, \alpha)$. It can be seen that it is well behaved and, despite the effects of noise and other distortions which introduce local maxima, the main peak remains distinct.

Let us conclude this paper with comments related to the rectilinearity measure derived by analysing the equation $\mathcal{P}_2(P) = \mathcal{P}_1(P, \alpha)$ which has a solution for some α if and only if P is a rectilinear polygon. A similar analysis[1] to those presented here led to the rectilinearity measure $\mathcal{R}_1(P)$ defined by

$$\mathcal{R}_1(P) = \frac{4}{4 - \pi} \cdot \left(\max_{\alpha \in [0, 2\pi]} \frac{\mathcal{P}_2(P)}{\mathcal{P}_1(P, \alpha)} - \frac{\pi}{4} \right).$$

If we consider the third polygon on the second row and the third polygon on the third row from Fig. 4, we see that there exist polygons P and Q such that

$$\mathcal{R}(P) < \mathcal{R}(Q) \qquad \text{and} \qquad \mathcal{R}_1(P) > \mathcal{R}_1(Q).$$

so, we can conclude that rectilinearity measures $\mathcal{R}$ and $\mathcal{R}_1$ are essentialy different even though they are derived in a similar manner. That is, there exists no monotonic function f, such that $f(\mathcal{R}(P)) = \mathcal{R}_1(P)$ holds for any polygon.

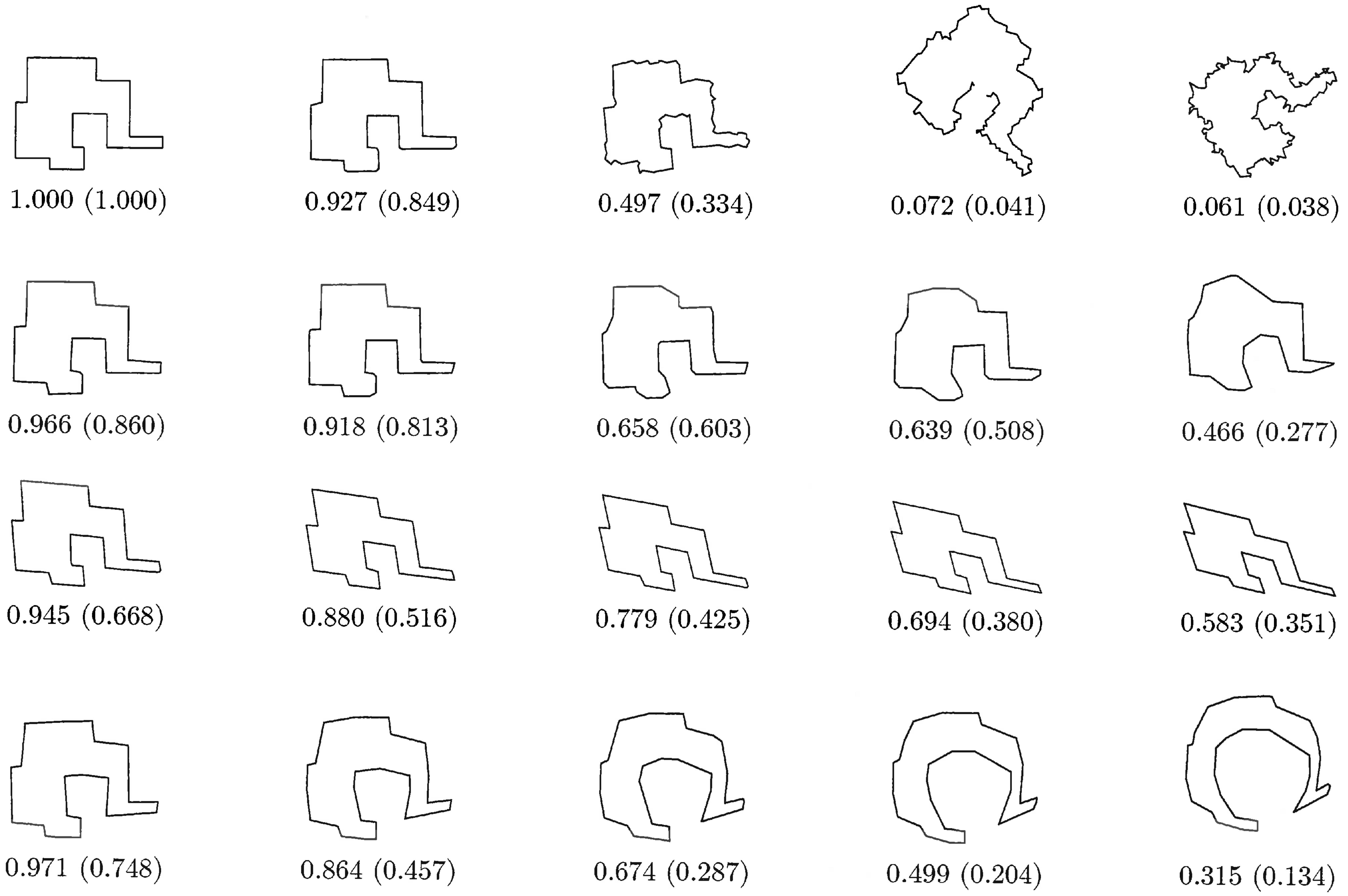

Figure 4. Examples of polygons with their rectilinearity measured as proposed in this paper. Polygons are rotated to the orientations plus 45° that maximised $\mathcal{Q}(P, \alpha)$. In the brackets are given rectangularity values measured by $\mathcal{R}_1$.[1]

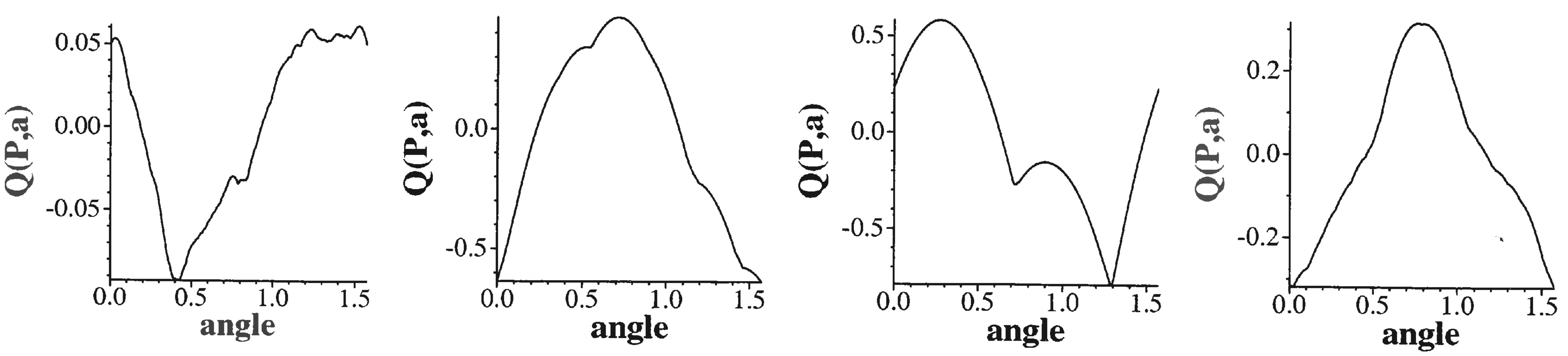

Figure 5. Plots of normalised $\mathcal{Q}(P, \alpha)$ for each of the rightmost examples in Fig. 4.

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